Splines, lattice points, and (arithmetic) matroids

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Outline

1. Examples
2. Definitions and results

Frobenius' coin exchange problem

Question

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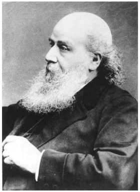
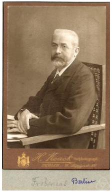
James Joseph Sylvester (1814–1897)



Ferdinand Georg Frobenius (1849-1917)



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A formula

- We are interested in finding non-negative integers a, b s. t. $2a + 3b = n$.

$$i_{(2,3)}(n) = \#\{(a,b) : 2a + 3b = n\}$$

$$= \begin{cases} \frac{n}{6} + 1 & n \equiv 0 \mod 6 \\ \frac{n-1}{6} & n \equiv 1 \mod 6 \\ \frac{n-2}{6} + 1 & n \equiv 2 \mod 6 \\ \frac{n-3}{6} + 1 & n \equiv 3 \mod 6 \\ \frac{n-4}{6} + 1 & n \equiv 4 \mod 6 \\ \frac{n-5}{6} + 1 & n \equiv 5 \mod 6 \end{cases}$$

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&= \frac{n}{6} + \frac{5}{12} + (-1)^n \frac{1}{4} + \xi_3^n \frac{1}{6}\left(1 + \frac{1}{\sqrt{3}} i\right) + \xi_3^{2n} \frac{1}{6}\left(1 - \frac{1}{\sqrt{3}} i\right)
\end{aligned}
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where $\xi_3 := e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3} i)$ (root of unity).

Geometry of the coin exchange problem

- We are interested in finding non-negative integers a, b s. t. $2a + 3b = n$.
- This corresponds to integer points in the polytope $\Pi_{(2,3)} = \{(a, b) \in \mathbb{R}^2_{\geq 0} : 2a + 3b = n\}$.
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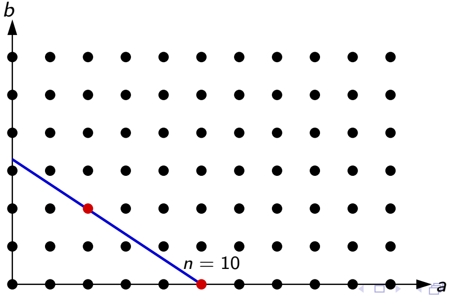

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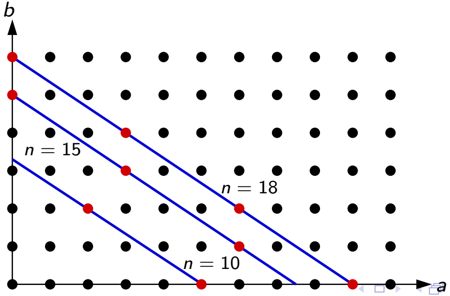
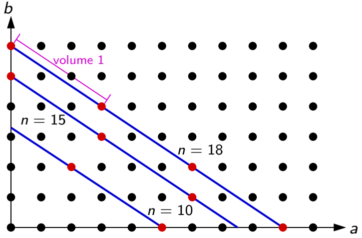


Volume and the number of integer points



- normalised volume: $T_{(2,3)}(n) := \mathrm{vol}_1(\Pi_{(2,3)}(n)) = \frac{n}{6}$.
- $i_{(2,3)}(n) = \frac{n}{6} + \frac{5}{12} + (-1)^n \frac{1}{4} + \xi_3^n(1 + \frac{1}{\sqrt{3}} i) + \xi_3^{2n}(1 - \frac{1}{\sqrt{3}} i)$

Remark

- $T_{(2,3)}(n) \approx i_{(2,3)}(n)$
- $i_{(2,3)}$ *can be obtained by applying the following differential operator to* $T_{(2,3)}(n)$: $\mathrm{Todd}(2,3)(\frac{\partial}{\partial n}) = 1 + \frac{5}{2}\frac{\partial}{\partial n} + \xi_2^n \frac{3}{2}\frac{\partial}{\partial n} + \xi_3^n \frac{2}{1-\xi_3}\frac{\partial}{\partial n} + \xi_3^{2n} \frac{2}{1-\xi_3^2}\frac{\partial}{\partial n}$

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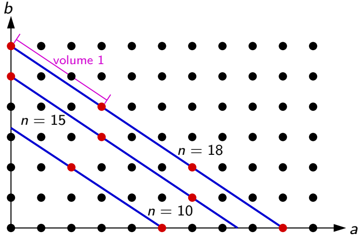

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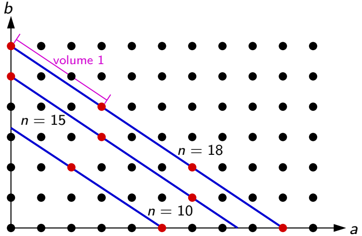

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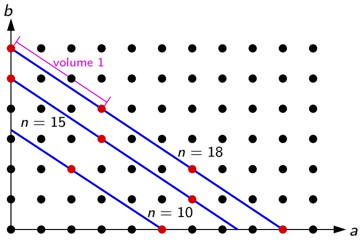

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Vector partition functions

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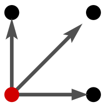
Let $(u, v) \in \mathbb{Z}^2$. In how many different ways can we write (u, v) as a sum of $(1, 0)$, $(0, 1)$, $(1, 1)$?



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We are interested in the integer points in the polytope $\Pi_X(u, v) = \{(a, b, c) \in \mathbb{R}^3_{\geq 0} : a \cdot (1, 0) + b \cdot (0, 1) + c \cdot (1, 1) = (u, v)\}$.

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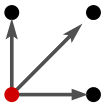

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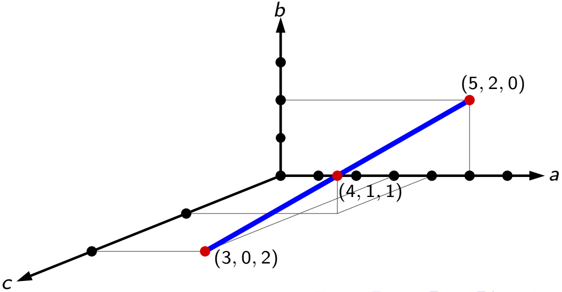

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- $i_X : \mathbb{Z}^2 \to \mathbb{N}_0$ *assigns to* (u,v) *the number of integer points in the polytope* $\Pi_X(u,v)$
- $T_X : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ *assigns to* (u,v) *the normalised volume of the polytope* $\Pi_X(u,v)$

Vector partition functions

- $i_X(u,v) = \begin{cases} \min(u+1, v+1) & u,v \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- $T_X(u,v) = \begin{cases} \min(u, v) & u,v \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- $\mathrm{Todd}(X)(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = 1 + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ transforms T_X into i_X.
- $\mathrm{Todd}(X)(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) T_X$ is not always well-defined.
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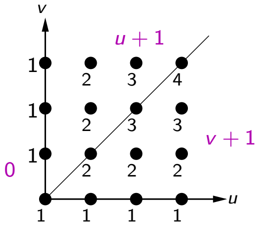

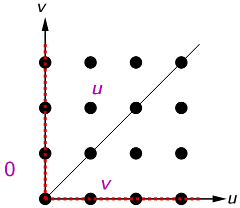

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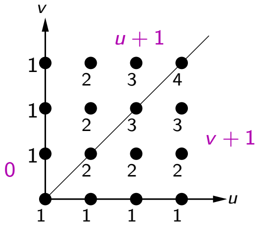

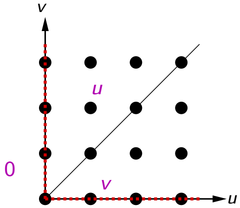

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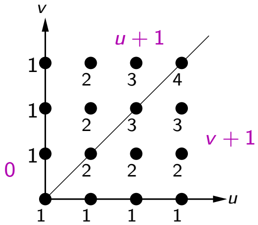
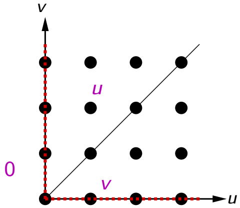


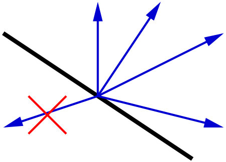


Setup

- $X = (x_1, \dots, x_N) \subseteq \mathbb{Z}^d$, list of vectors / $(d \times N)$-matrix
- $d \leq N$, full rank
- $0 \notin \mathrm{conv}(x_1, \dots, x_N)$.
- Sometimes we assume: X unimodular, i. e. every non-singular $(d \times d)$ submatrix has determinant $\pm 1 \Leftrightarrow$ every $\mathbb{R}^d$ basis selected from X is a lattice basis for $\mathbb{Z}^d$.

Example

- $X = (2, 3)$
- $X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$



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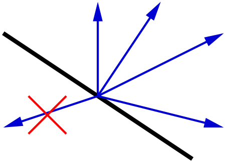
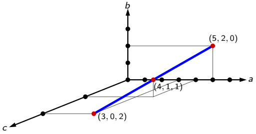


Variable polytopes

Definition (Variable polytopes)

Let $u \in \mathbb{R}^d$.

$$\Pi_X(u) := \{\alpha \in \mathbb{R}^N_{\geq 0} : X\alpha = u\}$$



Definition

We define the multivariate spline $T_X : \mathbb{R}^d \to \mathbb{R}$ and the vector partition function $i_X : \mathbb{Z}^d \to \mathbb{N}_0$ by

$$T_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \operatorname{vol}_{N-d}(\Pi_X(u))$$

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- *i_X generalizes the Ehrhart polynomial.*

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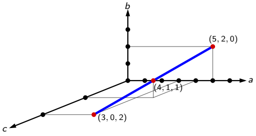

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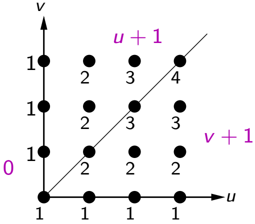

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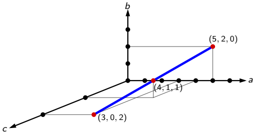

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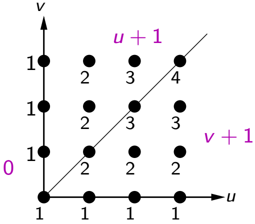

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Properties

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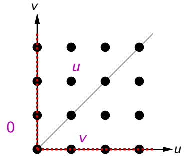
1. $\mathrm{supp}(T_X) = \left\{ \sum_{i=1}^{N} \lambda_i x_i : \lambda_i \geq 0 \right\} =: \mathrm{cone}(X)$
2. *T_X is piecewise polynomial of degree $N - d$.*
3. *Regions of polynomiality are cones*
4. *i_X is piecewise quasipolynomial (quasipolynomial: agrees with a polynomial on each coset of some sublattice $\Gamma \subseteq \mathbb{Z}^d$)*
5. *i_X is piecewise polynomial if X is unimodular*



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- $x = (v_1, \ldots, v_d) \in \mathbb{R}^d \rightsquigarrow p_x := v_1 s_1 + \ldots + v_d s_d \in \mathbb{R}[s_1, \ldots, s_d]$
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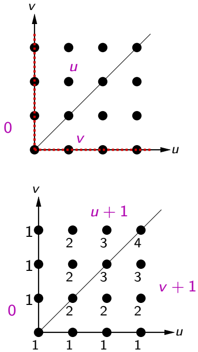

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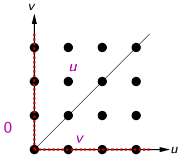
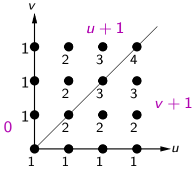
5. *i_X is piecewise polynomial if X is unimodular*





Notation

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- For example, $x = (1, 2) \rightsquigarrow p_x = s_1 + 2 s_2$.

Khovanskii-Pukhlikov formula

Definition

Todd operator: $\mathrm{Todd}(X) := \prod_{x \in X} \frac{p_x}{1-e^{-p_x}} \in \mathbb{R}[[s_1, \ldots, s_d]]$

Remark

- $\frac{p_x}{1-e^{-p_x}} = \sum_{k \geq 0} \frac{B_k}{k!} (p_x)^k$
- *B_i denote the Bernoulli numbers* $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$

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- *Suppose X unimodular.*
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Zonotopes

Remark

- T_X *has degree* $N - d$, *so* $\mathrm{Todd}(X)$ *is "too long".*
- *A differential operator of degree at most* $N - d$ *should be sufficient.*

Zonotopes

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Definition

- zonotope: $Z(X) := \{\sum_{i=1}^{N} \lambda_i x_i : 0 \leq \lambda_i \leq 1\} = X \cdot [0,1]^N$
- $\mathcal{Z}_-(X) :=$ set of interior lattice points in zonotope $Z(X)$

Example

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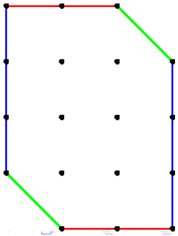

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Definition (Akopyan-Saakyan, 1988, Dyn-Ron, 1990)

central $\mathcal{P}$-space: $\mathcal{P}(X) := \operatorname{span}\{p_Y : Y \subseteq X, \operatorname{rank}(X \setminus Y) = \operatorname{rank}(X)\}$

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Internal $\mathcal{P}$-space

Remark

Todd$(X)(D)T_X$ *is not everywere well defined.*

Definition (Holtz-Ron, 2011, (Ardila-Postnikov, 2010))

$$\text{internal } \mathcal{P}\text{-space } \mathcal{P}_-(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x)$$

Example

- $X = (1, 1, 1)$
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Theorem (ML)

Let $p \in \mathcal{P}(X)$. Then $p \in \mathcal{P}_-(X) \Leftrightarrow p(D)T_X$ is continuous.

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Variant of Khovanskii-Pukhlikov

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Let $z \in \mathbb{R}^d$. Then define

$$f_z := \psi_X(e^{-p_z}\,\mathrm{Todd}(X))$$

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Suppose X unimodular. Let $z \in \mathcal{Z}_-(X)$. Then $f_z \in \mathcal{P}_-(X)$.

Theorem (ML, Variant of Khovanskii-Pukhlikov)

Suppose X unimodular. Let $u \in \mathbb{Z}^d$ and $z \in \mathcal{Z}_-(X)$. Then

$$f_z(D)T_X(u) = i_X(u - z).$$

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Suppose X unimodular. Then $\{f_z : z \in \mathcal{Z}_-(X)\}$ is a basis for $\mathcal{P}_-(X)$.

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Matroids and Hilbert series

- Hilbert series: $V = \bigoplus V_i$ graded vector space $\rightsquigarrow \mathrm{Hilb}(V, q) = \sum_i \dim V_i q^i$.
- Representation of a Matroid: a tuple (X, Δ) with X a list of vectors and $\Delta \subseteq 2^X$ linearly independent sublists
- Tutte polynomial: $\mathfrak{T}_X(\alpha, \beta) := \sum_{A \subseteq X} (\alpha - 1)^{r - \mathrm{rank}(A)} (\beta - 1)^{|A| - \mathrm{rank}(A)}$

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- $\mathrm{Hilb}(\mathcal{P}_-(X), q) = q^{N-d} \mathfrak{T}_X(0, q^{-1})$
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What can we do if X is not unimodular?

Toric arrangements

Definition

$x \in \mathbb{R}^d$ defines the following:

- in $\mathbb{R}^d$ the hyperplane $H_x = \{v \in \mathbb{R}^d : v \cdot x = 0\}$
- in $(\mathbb{R}/\mathbb{Z})^d$ the hypersurface $H_x^t = \{\phi \in (\mathbb{R}/\mathbb{Z})^d : \phi \cdot x = 0\}$

$\{H_x^t : x \in X\}$ is the toric arrangement defined by X.

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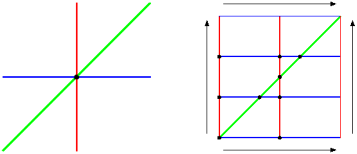

- $\mathcal{V}(X)$ = vertices of the toric arrangement
- $\phi \in \mathcal{V}(X)$ defines a map $e_\phi : \mathbb{R}^d \to S^1 \subseteq \mathbb{C}$ by $e_\phi(x) := e^{2\pi i(\phi \cdot x)}$
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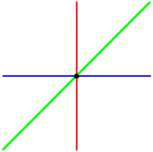

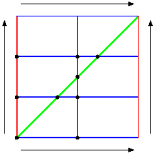

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Periodic Todd operator and the Brion-Vergne formula

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$$\widetilde{\text{Todd}}(X) := \sum_{\phi \in \mathcal{V}(X)} e_\phi \prod_{x \in X} \frac{p_x}{1 - e_\phi(-p_x) e^{-p_x}}$$

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End of talk

References

- Unimodular case: ML, *Lattice points in polytopes, box splines, and Todd operators*, IMRN, 2014
  `arXiv:1305.2784`, DOI: `10.1093/imrn/rnu095`
- General case: ML, *Splines, lattice points, and arithmetic matroids*, in preparation.

Appendix

Definition (Cocircuit ideal and projection)

- $\mathcal{J}(X) := \text{ideal}\{p_C : C \subseteq X \text{ and } \text{rank}(X \setminus C) < d\} \subseteq \mathbb{R}[s_1, \ldots, s_d]$.
- It is known that $\mathcal{P}(X) \oplus \mathcal{J}(X) = \mathbb{R}[s_1, \ldots, s_d]$.
- Let $\psi_X : \mathcal{P}(X) \oplus \mathcal{J}(X) \to \mathcal{P}(X)$ denote the projection.

Projections

- Unimodular case: $\psi_X : \mathbb{R}[[s_1, \ldots, s_d]] \to \mathcal{P}(X)$ projection map
- General case: $\widetilde{\psi}_X : \bigoplus_{\phi \in \mathcal{V}(X)} e_\phi \mathbb{R}[[s_1, \ldots, s_d]] \to \widetilde{\mathcal{P}}(X)$
- Let $f = \sum_{\phi \in \mathcal{V}(X)} e_\phi f_\phi \in \bigoplus_{\phi \in \mathcal{V}(X)} e_\phi \mathbb{R}[[s_1, \ldots, s_d]]$ ($f_\phi \in \mathbb{R}[[s_1, \ldots, s_d]]$ for all $\phi \in \mathcal{V}(X)$)
- $\widetilde{\psi}_X(f) := \sum_\phi e_\phi \psi_X(f_\phi)$
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