A convolution formula for Tutte polynomials of arithmetic matroids and other combinatorial structures

1. Matroids

Definition. A matroid is a pair \((M, rk)\), where
- \(M\) finite set (ground set)
- \(rk : 2^M \rightarrow \mathbb{Z}_{\geq 0}\) rank function satisfies axioms:
  - \(0 \leq rk(A) \leq |A|\)
  - \(A \subseteq B \implies rk(A) \leq rk(B)\)
  - \(rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)\)
- \(rk\) is a matroid

2. Arithmetic matroids

Definition (Bränden, D’Adderio, Moci).
- An arithmetic matroid (AM) is a triple \((M, rk, m)\)
  - \((M, rk)\) a matroid
  - \(m : 2^M \rightarrow \mathbb{Z}\) a multiplicity function that satisfies:
    - For \(A \subseteq M\), \(m(A) = \frac{\sum_{i \in A} m_i}{}\) for some \(m_i\) that sum over...
- **Tutte polynomial:**
  \[ \tau_M(x, y) := \sum_{A \subseteq M} m(A)(-1)^{|A| - rk(A)}(x - 1)^{|A| - rk(A)} \]

3. Further generalizations

The convolution formula holds in a more general setting:
- Relax the arithmetic matroid axioms
- Use different multiplicity functions

A ranked set with multiplicities is a finite set \(M\) together with
- A rank function \(rk : 2^M \rightarrow \mathbb{Z}\), \(rk(\emptyset) = 0\)
- A function \(m : 2^M \rightarrow R\), \(R \) is a commutative ring with 1.
- Deletion and contraction are defined in the usual way.

Theorem (ML [3]). Let \((M, rk, m)\) and \((M, rk, m)\) be two ranked sets with multiplicity. Then
\[ \tau_M(1, p, 1-q) = \tau_N(1, p, 1-q) \tau_N(p, q) \]
for infinitely many \(p, q \in \mathbb{Z}\).

4. Applications

New proofs of two known positivity results:
- The coefficients of the arithmetic Tutte polynomial are positive
- The product of the multiplicity functions of two AMs is again a multiplicity function of an AM.

In special cases, we can recover known results:
- Let \(X = (x_1, \ldots, x_d) \subseteq \mathbb{Z}^d\).
- For \((x, y) = (2, 1)\), our theorem is equivalent to a well-known identity for the zonotope \(Z(X) = \sum_{i=1}^d \mathbb{X}(x_i) \quad 0 \leq i \leq d\):
  \[ |Z(X) \cap \mathbb{Z}^d| = \binom{m(1, 0)}{0} \sum_{A \subseteq X} rk(A) \]
  where the last sum is over all faces of \(X\).

5. Flows and colourings

- \(X = (x_1, \ldots, x_d) \subseteq \mathbb{Z}^d\)
- \(\phi \in \text{Hom}(\mathbb{Z}^d, \mathbb{Z}/q\mathbb{Z})\) proper arithmetic q-coloring if \(\phi(x) \neq 0\) for all \(x \in X\)
- A nowhere zero q-flow on \(X\) is \(\phi : X \rightarrow \mathbb{Z}/q\mathbb{Z} \setminus \{0\}\)
  \[ \sum_{x \in X} \phi(x) = 0 \implies \text{flow is external}\]
- \(\chi_k(x) := \text{no. of colourings and } \chi_k(x) := \text{no. of flows}\)
- \(\chi_k\) and \(\chi_k\) are quasi-polynomials
- Generalizes flows and colourings of graphs

This yields a combinatorial interpretation of the arithmetic Tutte polynomial at infinitely many points:
\[ \tau_N(1, p, 1-q) = \tau_N(1, p, 1-q) \tau_N(p, q) \]
for infinitely many \(p, q \in \mathbb{Z}\).

6. Powers of arithmetic matroids and Plücker coordinates

**Theorem (ML [3]).** \(\Phi \neq 1\) non-negative integer
- \(A = (m, rk, m)\) representable arithmetic matroid
- \((m, rk)\) non-regular
Then \(A^k = (m, rk, m)\) is not representable.

If \((m, rk)\) is regular and \(m\) satisfies an extra condition, then \(A^k\) is representable.

**Theorem (ML [3]).** \(k \neq 0\), \(k \neq 1\) non-negative integer
- \(X = (x_1, \ldots, x_d) \subseteq \mathbb{Z}^d\)
- \(X \in \mathbb{R}^d\)
- \(X\) is a \(k\)-regular matroid of full rank \(d \leq N\)

7. References


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