# A convolution formula for Tutte polynomials of arithmetic matroids and other combinatorial structures

## ① Matroids

**Definition.** A matroid is a pair $(M, \mathrm{rk})$, where
- $M$ finite set (*ground set*)
- $\mathrm{rk} : 2^M \to \mathbb{Z}_{\geq 0}$ *rank function* satisfies axioms:
  - $0 \leq \mathrm{rk}(A) \leq |A|$
  - $A \subseteq B \Rightarrow \mathrm{rk}(A) \leq \mathrm{rk}(B)$
  - $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$
- Tutte polynomial:

$$\mathfrak{T}_M(x, y) := \sum_{A \subseteq M} (x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$$

**Representable matroids:** A $(d \times N)$-matrix with entries in some field $\mathbb{K}$ defines a matroid in a canonical way:
- ground set: columns of the matrix
- rank function: rank function from linear algebra

**Theorem (Zaslavsky, 1975).** A hyperplane arrangement in $\mathbb{R}^d$ with corresponding matroid $(M, \mathrm{rk})$ divides $\mathbb{R}^d$ into $\mathfrak{T}_M(2, 0)$ regions.

- $(M, \mathrm{rk})$ matroid and $A \subseteq M$
  - restriction to $A$: $(M, \mathrm{rk})|_A := (A, \mathrm{rk}|_A)$
  - contraction of $A$: $(M, \mathrm{rk})/A := (M \setminus A, \mathrm{rk}_{/A})$ where $\mathrm{rk}_{/A} : 2^{M \setminus A} \to \mathbb{Z}_{\geq 0}$, $\mathrm{rk}_{/A}(S) := \mathrm{rk}(A \cup S) - \mathrm{rk}(A)$

**Theorem (KRS&ELV [2, 4], Matroid Convolution formula).** $(M, \mathrm{rk})$ matroid. Then

$$\mathfrak{T}_M(x, y) = \sum_{A \subseteq M} \mathfrak{T}_{M|_A}(0, y)\mathfrak{T}_{M/A}(x, 0).$$

## ② Arithmetic matroids

**Definition (Bränden, D'Adderio, Moci).**
- An arithmetic matroid (AM) is a triple $(M, \mathrm{rk}, m)$
  - $(M, \mathrm{rk})$ is a matroid
  - $m : 2^M \to \mathbb{Z}_{\geq 1}$ is the *multiplicity function* that satisfies:
    - For $A, B \subseteq M$, m(A) divides m(B) iff ...
    - $\sum_A (-1)^? m(A) \geq 0$, where we sum over ...
- Arithmetic Tutte polynomial:

$$\mathfrak{M}_M(x, y) := \sum_{A \subseteq M} m(A)(x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$$

**Representable AMs:** A $(d \times N)$-matrix with entries in $\mathbb{Z}$ defines an arithmetic matroid in a canonical way:
- defines a matroid in the usual way
- multiplicity of a basis $B$: $m(B) = |\det(B)|$
- in general: $m(S) := \left| \langle S \rangle_{\mathbb{R}} \cap \mathbb{Z}^d / \langle S \rangle_{\mathbb{Z}} \right|$

**Theorem (Lawrence, 2011).** A toric arrangement on the real torus $(S^1)^d$ with corresponding AM $(M, \mathrm{rk}, m)$ divides the torus into $\mathfrak{M}_M(1, 0)$ regions.

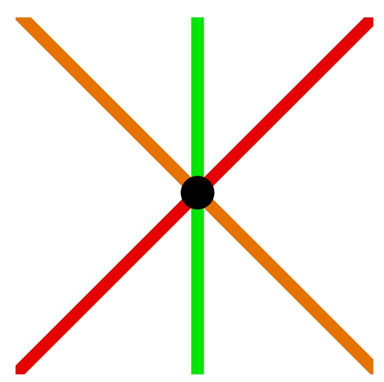
Restriction and contraction for the multiplicity function:
- $m|_A(S) = m(S)$ for $S \subseteq M$
- $m_{/A}(S) = m(A \cup S)$ for $S \subseteq M \setminus A$

**Theorem (ML–SB [1], Arithmetic convolution formula).** $(M, \mathrm{rk}, m)$ arithmetic matroid. Then

$$\mathfrak{M}_M(x, y) = \sum_{A \subseteq M} \mathfrak{M}_{M|_A}(0, y)\mathfrak{T}_{M/A}(x, 0) = \sum_{A \subseteq M} \mathfrak{T}_{M|_A}(0, y)\mathfrak{M}_{M/A}(x, 0).$$

## Hyperplane arrangements and toric arrangements

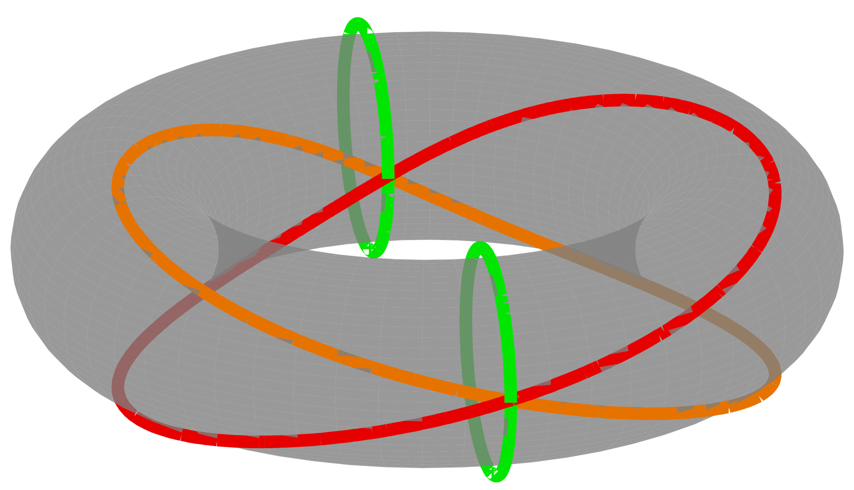
$$X = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$



hyperplane arrangement

related to volumes of polytopes

$\mathfrak{T}_X(x, y) = x^2 + x + y$



toric arrangement

related to no. of integer points in polytopes

$\mathfrak{M}_X(x, y) = x^2 + 2x + 2y + 1$

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## ③ Further generalizations

The convolution formula holds in a more general setting:
- relax the arithmetic matroid axioms
- use two different multiplicity functions

A ranked set with multiplicities is a finite set $M$ together with
- a rank function $\mathrm{rk} : 2^M \to \mathbb{Z}$ s. t. $\mathrm{rk}(\emptyset) = 0$
- a multiplicity function $m : 2^M \to R$, where $R$ is a commutative ring with 1.
- Deletion and contraction are defined in the usual way.

**Theorem (ML–SB [1]).** Let $(M, \mathrm{rk}, m_1)$ and $(M, \mathrm{rk}, m_2)$ be two ranked sets with multiplicity. Then $(M, \mathrm{rk}, m_1 \cdot m_2)$ is a ranked set with multiplicity and

$$\mathfrak{M}_{(M,\mathrm{rk},m_1 m_2)}(x, y) = \sum_{A \subseteq M} \mathfrak{M}_{(M,\mathrm{rk},m_1)|_A}(0, y)\mathfrak{M}_{(M,\mathrm{rk},m_2)/A}(x, 0).$$

This setting contains the following combinatorial structures:
- *Matroids*
- *(Pseudo-/quasi-) arithmetic matroids*
- *Integral polymatroids*: if rk is the submodular function that defines an integral polymatroid, $R = \mathbb{Z}$ and $m \equiv 1$
- *Rank functions of delta-matroids and ribbon graphs:* $m \equiv 1$ and $\mathrm{rk} = \rho$, the rank function of an even delta-matroid

  Ribbon graphs $\leftrightarrow$ delta-matroids like graphs $\leftrightarrow$ matroids

## ④ Applications

New proofs of two known positivity results:
- The coefficients of the arithm. Tutte polynomial are positive
- The product of the multiplicity functions of two AMs is again a multiplicity function of an AM.

In special cases, we can recover known results:
- Let $X = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^d$. For $(x, y) = (2, 1)$, our theorem is equivalent to a well-known identity for the zonotope $Z(X) := \{\sum_{i=1}^N \lambda_i x_i : 0 \leq \lambda_i \leq 1\}$:

$$\left| Z(X) \cap \mathbb{Z}^d \right| = \mathfrak{M}(2, 1) = \sum_{A \subseteq X} \mathfrak{M}_{M|_A}(0, 1)\mathfrak{T}_{M/A}(2, 0) = \sum_{X \supseteq A \text{ flat}} \mathfrak{M}_{M|_A}(0, 1)\mathfrak{T}_{M/A}(2, 0) = \sum_F \left| \mathrm{relint}(F) \cap \mathbb{Z}^d \right|,$$

  where the last sum is over all faces of $Z(X)$.
- setting $x = 1$ in convolution formula $\leftrightarrow$ decomposition $\mathrm{DM}(X) = \bigoplus_{p \in \mathcal{V}(X)} e_p \mathcal{D}(X_p)$ of Dahmen–Micchelli spaces appearing in the theory of vector partition functions

## ⑤ Flows and colourings

- $X = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^d$
- $\phi \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z}/q\mathbb{Z})$ *proper arithmetic q-coloring* if $\phi(x) \neq 0$ for all $x \in X$
- A *nowhere zero q-flow* on $X$ is $\psi : X \to \mathbb{Z}/q\mathbb{Z} \setminus \{0\}$ s. t. $\sum_{x \in X} \psi(x)x = 0$ in $\mathbb{Z}^d/q\mathbb{Z}^d$
- $\chi_X(q) :=$ no. of colourings and $\chi_X^*(q) :=$ no. of flows
- $\chi_X$ and $\chi_X^*$ are quasi-polynomials
- generalizes flows and colourings of graphs

This yields a combinatorial interpretation of the arithmetic Tutte polynomial at infinitely many points:

$$\mathfrak{M}_X(1-p, 1-q) = p^{\mathrm{rk}(G)-\mathrm{rk}(X)}(-1)^{\mathrm{rk}(X)} \cdot \sum_{A \subseteq X} (-1)^{|A|} \chi_{X|_A}^*(q)\chi_{X/A}(p)$$

for infinitely many $p, q \in \mathbb{Z}$.

## ⑦ Powers of arithmetic matroids and Plücker coordinates

**Theorem (ML [3]).**
- $k \neq 1$ non-negative integer
- $\mathcal{A} = (M, \mathrm{rk}, m)$ representable arithmetic matroid
- $(M, \mathrm{rk})$ non-regular

Then $\mathcal{A}^k := (M, \mathrm{rk}, m^k)$ is not representable.

If $(M, \mathrm{rk})$ is regular and $m$ satisfies an extra condition, then $\mathcal{A}^k$ is representable.

**Theorem (ML [3]).**
- $k \in \mathbb{R}_{\geq 0}$, $k \neq 1$
- $X \in \mathbb{R}^{d \times N}$ matrix of full rank $d \leq N$

Then $X$ represents a regular matroid if and only if the following condition is satisfied:

there is $X_k \in \mathbb{R}^{d \times N}$ s. t. for each maximal minor $\Delta_I(X)$, $|\Delta_I(X)|^k = |\Delta_I(X_k)|$ holds, where $I \in \binom{[N]}{d}$.

If $k$ is an integer, this holds over any ordered field $\mathbb{K}$.

## References

[1] SB and ML. A convolution formula for Tutte polynomials of arithmetic matroids and other combinatorial structures, 2016. arXiv:1602.02664.

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[3] ML. On powers of Plücker coordinates and representability of arithmetic matroids, 2017. arXiv:1703.10520.

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