# Interpolation, Box Splines, and Lattice Points in Zonotopes

## ① Main Result

- $X = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^d$ list of vectors equivalently, $X$ is a $(d \times N)$-matrix
- $N \geq d$
- $X$ is *totally unimodular* and has full rank.
- $\mathcal{Z}_-(X) := Z(X) \cap \mathbb{Z}^d$, interior lattice points of zonotope $Z(X)$
- $B_X$ box spline defined by $X$
- $p \in \mathbb{R}[s_1, \ldots, s_d] \rightsquigarrow$ differential operator $p(D) := p(\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_d})$

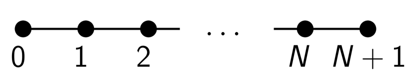
**Main Theorem [1]:** Let $f : \mathcal{Z}_-(X) \to \mathbb{R}$. There exists a unique polynomial $p \in \mathcal{P}_-(X) \subseteq \mathbb{R}[s_1, \ldots, s_d]$ s. t.
$$p(D)B_X|_{\mathcal{Z}_-(X)} = f.$$

- This was conjectured by Holtz and Ron [3].

**Proposition [1]:** Let $p \in \mathcal{P}_-(X)$. Then $p(D)B_X$ is a continuous function.

## The Main Theorem in $\mathbb{R}^1$

Let $X_{N+1} := (\underbrace{1, \ldots, 1}_{N+1 \text{ times}})$. Then

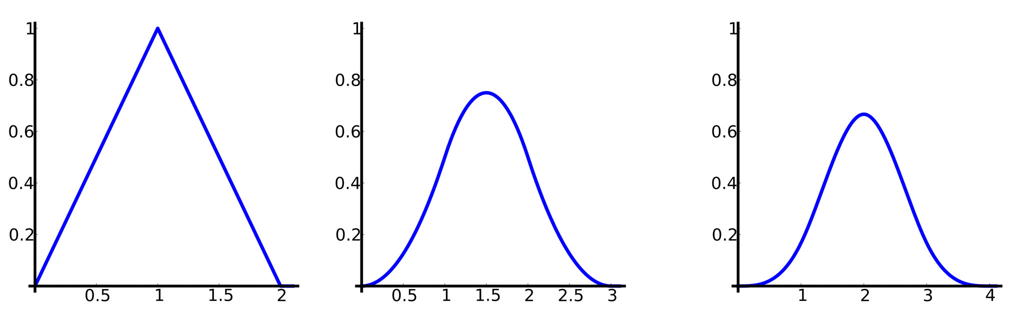



- $Z(X_{N+1}) = [0, N+1]$ and $\mathcal{Z}_-(X_{N+1}) = \{1, \ldots, N\}$
- $\mathcal{P}(X_{N+1}) = \operatorname{span}\{1, s, \ldots, s^N\}$
- $\mathcal{P}_-(X_{N+1}) = \operatorname{span}\{1, s, \ldots, s^{N-1}\}$

$$B_{X_{N+1}}(u) = \sum_{j=0}^{N+1} \frac{(-1)^j}{N!} \binom{N+1}{j} \max(u-j, 0)^N$$

Let $M^N$ be the $(N \times N)$-matrix given by
$$m^N_{ij} = D_x^{i-1} B_{X_{N+1}}(j) \quad \text{for } i, j \in \{1, 2, \ldots, N\}.$$

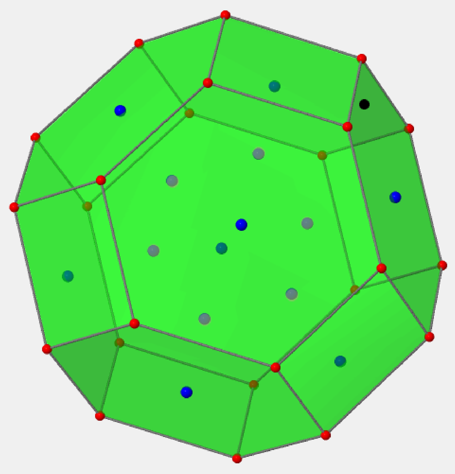
- The Main Theorem in $\mathbb{R}^1$ is equivalent to $M^N$ having fulling rank.



$$M^1 = (1) \qquad M^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \qquad M^3 = \begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & -2 & 1 \end{pmatrix}$$

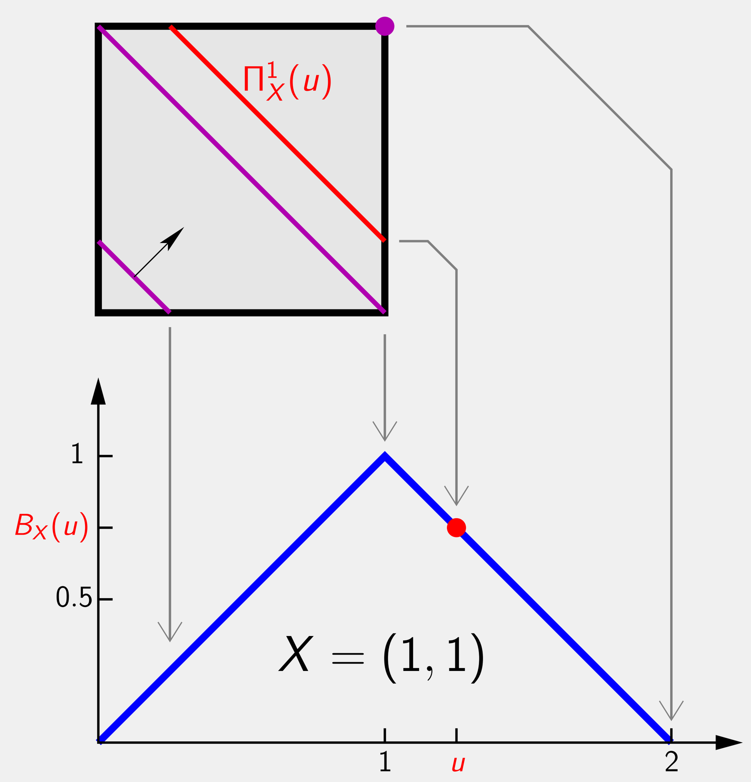
## ② Zonotope and Box Spline

Zonotope:
$$Z(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} \subseteq \mathbb{R}^d$$

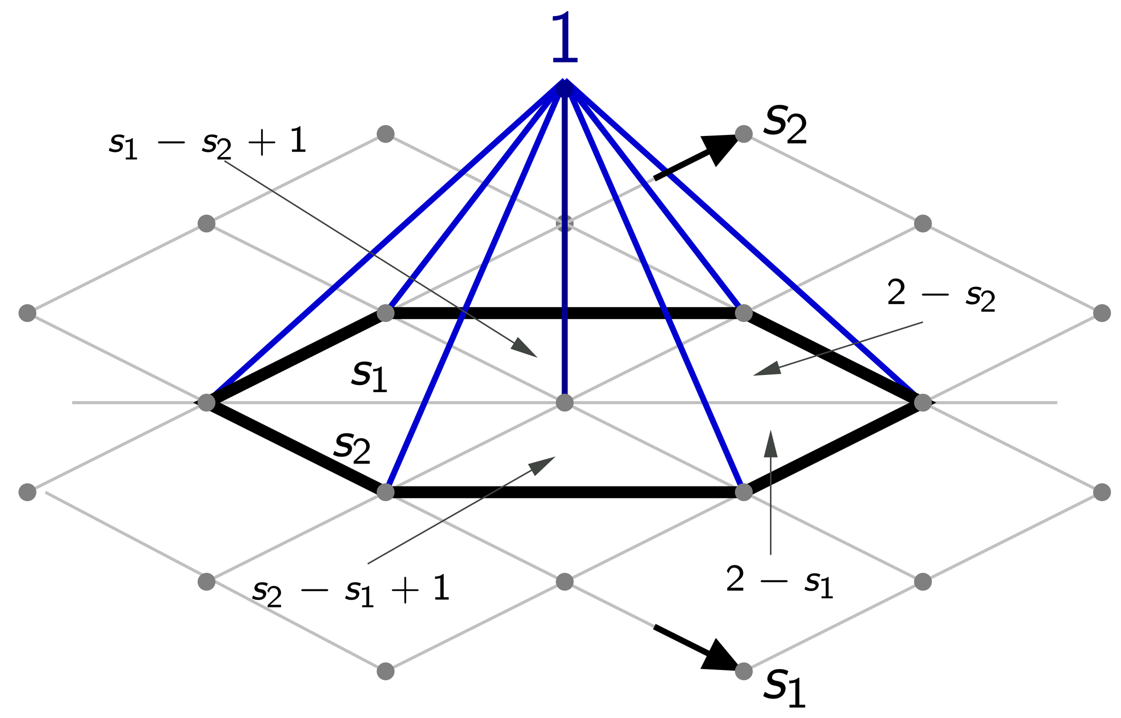


Slice of unit cube:
$$\Pi^1_X(u) := \{w \in [0,1]^N : Xw = u\}$$

Box spline: $B_X : \mathbb{R}^d \to \mathbb{R}$
$$B_X(u) := \frac{1}{\sqrt{\det(XX^T)}} \operatorname{vol}_{N-d} \Pi^1_X(u)$$




## An Example




$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathcal{P}(X) = \operatorname{span}\{1, s_1, s_2\}$$
$$\mathcal{Z}_-(X) = \{(1,1)\} \qquad \mathcal{P}_-(X) = \mathbb{R}$$

## ④ Some Facts

- $\dim \mathcal{P}(X) = \operatorname{vol}(Z(X))$
- $\dim \mathcal{P}_-(X) = \#$ interior lattice points in $Z(X)$.
- The Hilbert series of $\mathcal{P}(X)$ and $\mathcal{P}_-(X)$ are evaluations of the Tutte polynomial $T_X$ [5].
- $B_{(X,x)} = \int_0^1 B_X(u - \tau x)\, d\tau = B_X * B_x$
- $B_X$ is piecewise polynomial; local pieces have degree $N - d$
- For $x \in X$, $D_x B_X = \nabla_x B_{X \setminus x} := B_{X \setminus x} - B_{X \setminus x}(\cdot - x)$

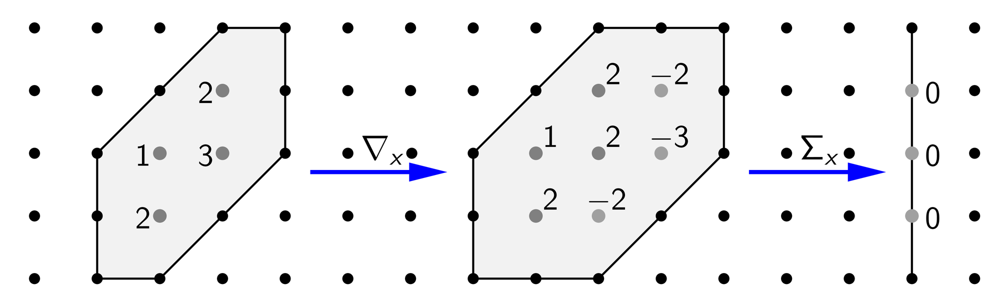
## ③ $\mathcal{P}$-spaces

- $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ $\rightsquigarrow p_v := v_1 s_1 + \ldots + v_d s_d \in \mathbb{R}[s_1, \ldots, s_d]$.
- $Y \subseteq X$ sublist $\rightsquigarrow p_Y := \prod_{y \in Y} p_y$.
- e. g. $Y = ((1,0),(1,2))$ $\rightsquigarrow p_Y = s_1(s_1 + 2s_2) = s_1^2 + 2s_1 s_2$
- central $\mathcal{P}$-space: $\mathcal{P}(X) := \operatorname{span}\{p_Y : \operatorname{rank}(X \setminus Y) = \operatorname{rank}(X)\}$
- internal $\mathcal{P}$-space [3, 5]: $\mathcal{P}_-(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x)$

## Proof: Deletion-Contraction

- $X \setminus x$ *deletion*.
- $\pi_x : \mathbb{R}^d \to \mathbb{R}^d / \operatorname{span}(x)$ canonical projection
- $X/x$ *contraction*: image of $X \setminus x$ under $\pi_x$
- $X \setminus x$ and $X/x$ represent deletion and contraction of the matroid defined by $X$.

$$\Xi(X) := \{f : \mathbb{Z}^d \to \mathbb{R} : \operatorname{supp}(f) \subseteq \mathcal{Z}_-(X)\}$$
$$\gamma_X : \mathcal{P}_-(X) \to \Xi(X)$$
$$p \mapsto p(D)B_X|_{\mathbb{Z}^d}$$




**Proposition [1]:** ▶ $x \in X$ neither a loop nor a coloop. Then
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{P}_-(X \setminus x) & \xrightarrow{\cdot p_x} & \mathcal{P}_-(X) & \xrightarrow{\pi_x} & \mathcal{P}_-(X/x) & \to & 0 \\ & & \downarrow \gamma_{X \setminus x} & & \downarrow \gamma_X & & \downarrow \gamma_{X/x} & & \\ 0 & \to & \Xi(X \setminus x) & \xrightarrow{\nabla_x} & \Xi(X) & \xrightarrow{\Sigma_x} & \Xi(X/x) & \to & 0 \end{array}$$
is exact and the vertical maps are isomorphisms.

- $\nabla_x(f)(z) := f(z) - f(z - x)$
- $\Sigma_x(f)(\bar{z}) := \sum_{x \in \bar{z} \cap \mathbb{Z}^d} f(x)$

- This proposition is the central part of the inductive proof of the Main Theorem.

## ⑤ Outlook: Deriving the Khovanskii-Pukhlikov formula

- For every $z \in \mathcal{Z}_-(X)$, there exists a unique $f_z \in \mathcal{P}_-(X)$ s. t. $f_z(D)B_X|_{\mathbb{Z}^d} = \delta_z$ (Main Theorem).

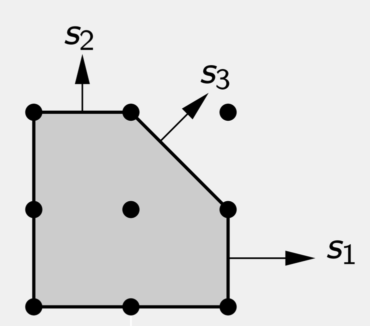
**Theorem [2]:** $f_z$ is the projection to $\mathcal{P}(X)$ of
$$\operatorname{Todd}(X, z) := e^{-z} \prod_{x \in X} \frac{x}{1 - e^{-x}} \in \mathbb{R}[[s_1, \ldots, s_d]].$$

- $\Pi_X(u) := \{w \in \mathbb{R}^N_{\geq 0} : Xw = u\}$ polytope

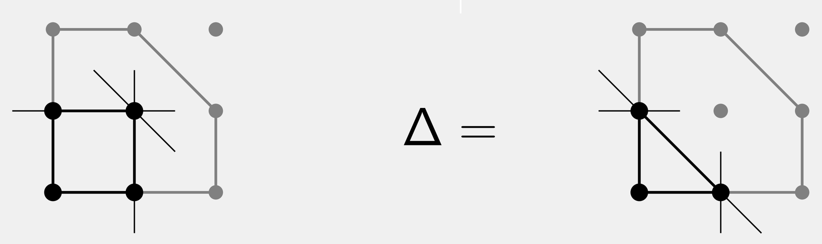
**Corollary [2] (Khovanskii-Pukhlikov):**
- $X \in \mathbb{Z}^{d \times N}$ totally unimodular and full rank
- Let $z \in \mathcal{Z}_-(X)$ and $u \in \mathbb{Z}^d$.

Then $|\Pi_X(u - z) \cap \mathbb{Z}^d| = f_z(D) \operatorname{vol}(\Pi_X(u))$.

$\Pi_X(2,2,3) = ⬠ =$




$\square =$ $\Delta =$



$$|\square \cap \mathbb{Z}^2| = \operatorname{vol}(⬠) + \frac{1}{2} \frac{\partial}{\partial s_3} \operatorname{vol}(⬠) = 4$$
$$\text{and } |\Delta \cap \mathbb{Z}^2| = \operatorname{vol}(⬠) - \frac{1}{2} \frac{\partial}{\partial s_3} \operatorname{vol}(⬠) = 3.$$

## References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